03/19/11

James Cogdell – Local Exterior square

L-function for $GL(n)$

where $n = 2m$

I. Integral Representations

k = local non-Archimedean.

Jacquet–Shalika (1990) Let π be an irred admissible generic rep'n of $GL_n(k)$. Realize it in its Whittaker model, $W \in \mathcal{W}(\pi, \psi)$.

Let $\Phi \in C_c^\infty(k^m)$.

$$J(s, W, \Phi) = \int_{N_m \backslash GL_m} \int_{V_m \backslash M_m} W\left(\sigma\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi(\mathrm{trace}(x))\, dx$$

times

$$\Phi(e_m g)\, |\det(g)|^s\, dg$$

where $e_n = (0, 0, \ldots, 1)$ and

$$\sigma = \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n \\ 1 & 3 & \cdots & n-1 & 2\ 4 & \cdots & n \end{pmatrix}$$

Integral converges for

$\mathrm{Re}(s) \gg 0$ and $\mathrm{Re}(s) > 1$??

<u>Theorem</u> (Sacquet–Shalika)

(i) $J(s, W, \Phi) \in \mathbb{C}(q^{-s})$, so you have meromorphic ~~continuation~~ continuation.

(ii) $\langle J(s, W, \Phi) \rangle$ is a fractional $\mathbb{C}[q^s, q^{-s}]$-ideal in $\mathbb{C}(q^{-s})$.

$$\langle J(s, W, \Phi) \rangle = \left(\frac{1}{(P(q^{-s}))} \right)$$

$$P(0) = 1.$$

(iii) Local functional equation

$$J(s, W, \Phi) = \gamma(s, \pi, \Lambda^2, \psi)\, J(s, W, \Phi)$$

(iv) If $\pi, \psi, k/\mathbb{Q}_p$ are all unramified, then

$$J(s, W, \underline{\Phi}) = L(s, \pi, \Lambda^2).$$

<u>Defn:</u> At the ramified places, define

$$L(s, \pi, \Lambda^2) = \frac{1}{P(q^{-s})}$$

<u>Question:</u> So what really happens at the ramified places? What if the field is Archimedean??

<u>Ramified places:</u> $\mathcal{S}$ Schwartz functions $= \mathcal{S}(k^m)$

$$\mathcal{S}(k^m) \supset \mathcal{S}_0(k^m) \supset \{0\}$$

$$\mathcal{S}_0(k^m) = \{\underline{\Phi} \in \mathcal{S}(k^m) : \underline{\Phi}(0) = 0\}$$

<u>Defn</u>: Let s_0 be a ~~pole~~ pole of $J(s, W, \underline{\phi})$. So

$$J(s, W, \underline{\phi}) = \frac{B_{s_0}(W, \underline{\phi})}{(q^s - q^{s_0})^k} + \cdots$$

Then we say that s_0 is an exceptional pole if $B_{s_0}(W, \underline{\phi})$ vanishes on $\mathcal{S}_0(h^m)$.

In this case, $B_{s_0}(W, \underline{\phi}) = \underline{\phi}(0)\Lambda_{s_0}(W)$

Λ_{s_0} is a twisted Shalika functional, so

$$\Lambda_{s_0}\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix} W\right) = \psi(\mathrm{trace}(x))\,|\det(g)|^s \Lambda_{s_0}(W)$$

Space of Shalika functionals is 1-dimensional. If π supports a non-trivial Shalika functional, then $\pi = \tilde{\pi}$.

OTOH, Bernstein–Zelevinsky analyzed the structure of rep'ns GL_n by $\pi|_{P_n}$ where P_n is the Mirabolic subgroup. They got

$\{\pi^{(n-k)}\}$, a system of derivatives of π.

$$\pi^{(n-k)} \in \widehat{GL_k}$$

<u>Theorem</u>: If $\pi^{(k)}$ are all completely reducible, i.e. $\pi^{(k)} = \bigoplus \pi_j^{(k)}$,

Then $L(s, \pi, \Lambda^2) = \operatorname{lcm}_{k,j} L_{ex}(s, \pi_j^{(2k)}, \Lambda^2)$

exceptional poles.

<u>Question</u>: Does this work for Archimedean places? (student of Cogdell is working on this)

P. Kewat (student of Raghuram) uses Bernstein-Zelevinsky derivatives to understand

$$W\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

Bernstein-Zelevinsky

(asymptotics of $W \longleftrightarrow$ exponents of BZ derivatives of π)

<u>Thm</u>: If π is square-integrable, then $J(s, W, \Phi)$ converge for $Re(s) > 0$ and has a pole at $s = 0$ iff a certain Shalika functional is non-zero.

<u>Thm</u>: If $Re(s_0) > 0$, $\exists$ data W_{s_0}, Φ_{s_0}

s.t. $J(s_0, W_{s_0}, \Phi_{s_0}) \neq 0$.

Dustin Belt (Purdue) proved:

<u>Theorem</u>: Let k be a local field (Archimedean or non-Archimedean). Given $s_0 \in \mathbb{C}$, $\exists$ data W_{s_0}, Φ_{s_0} such that

$$J(s_0, W_{s_0}, \Phi_{s_0}) \neq 0.$$

<u>Theorem</u>: Let k be a number field and π a cuspidal automorphic rep'n of GL_n.

Then $L^S(s, \pi, \Lambda^2)$ extends to a meromorphic continuation, entire if $\omega_\pi \neq 1$.

If $\omega_\pi = 1$, has a simple pole at $s = 0, 1$.

partial L-function.

Langlands – Shahidi Method :

Let π be an irred. admissible rep'n of

$GL_n(k)$. Let $GL_n \cong M \subset P = MN \subset SO_{2n}(k)$

Siegel Parabolic

The rep'n of ${}^LM$ on $Lie({}^LN)$ is

Λ^2 of GL_n.

Local Coefficient :

$$c_{\psi^{-1}}(s, \tilde{\pi}) = \gamma(s, \pi, \Lambda^2, \psi)$$

$$= \frac{\varepsilon(s, \pi, \Lambda^2, \psi)\, L(1-s, \tilde{\pi}, \Lambda^2)}{L(s, \pi, \Lambda^2)} \quad *$$

Shahidi, in his work, some how understands $c_{\psi^{-1}}(s, \tilde{\pi})$. Therefore, he is able to understand L-functions by $*$

Question: Stability of γ under highly ramified twists. i.e. what can we say about

$\gamma(s, \pi \otimes \eta, \Lambda^2, \psi)$ where η is highly ramified??

T. L. Tsai (student of Shahidi)
understands stability of γ for
Rankin-Selberg for $GL_n \times GL_n$.

Combine ideas of Tsai with γ = Mellin transform of Bessel function,

then get

<u>Thm</u>: Let π_1, π_2 be s.c. rep'ns of GL_n with the same central characters, then for sufficiently highly ramified characters η of $k^\times$, get

$$\gamma(s, \pi_1 \otimes \eta, \Lambda^2, \psi) = \gamma(s, \pi_2 \otimes \eta, \Lambda^2, \psi)$$

i.e. get stability of γ-factors

Combine this with some local/global techniques, and stability of γ for Galois representations, given by Deligne-Henniart, and with multiplicity of γ, then get the following:

<u>Theorem</u>: Let π be an irred admissible generic rep'n of GL_n. Let $\rho = \rho(\pi)$ be the corresponding rep'n of W_k' under LLC. Then

$$\gamma(s, \pi, \Lambda^2, \psi) = \gamma(s, \Lambda^2(\rho), \psi).$$

Since Henniart proved that

$$L(s, \pi, \Lambda^2) = L(s, \Lambda^2(\rho)),$$ we get

$$\varepsilon(s, \pi, \Lambda^2, \psi) = \varepsilon(s, \Lambda^2(\rho), \psi).$$